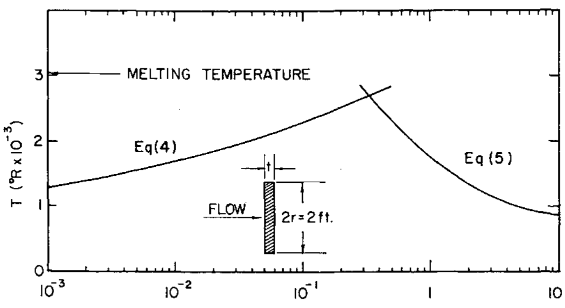

Fig. 1 Calculated maximum temperature T_w of a steel disk of thickness t during re-entry from a decaying earth orbit

steel over the expected temperature range. A value for total emissivity of $\epsilon = 0.6$ was chosen from Ref. 5 for high-temperature steel.

For the particular case of a steel disk, the parameter W/C_DA reduces to

$$(W/C_DA) = (\rho At/C_DA) = 21t \tag{3}$$

in the units given in the Nomenclature. In evaluating Eqs. (1) and (2) it has been assumed that heat is radiated from both sides of the disk, that $R = 4r$ (cf., Ref. 6), and that the initial temperature of the fragment is 500°R. The equations for the maximum temperature of the fragment during re-entry then reduce to

$$T_w = 3070t^{1/8} \tag{4}$$

and

$$T_w = 500 + 1230t^{-1/2} \tag{5}$$

The resulting curves for T_w as a function of t are given in Fig. 1, which shows that the temperature for both the very thin and very thick fragments (corresponding to low and high values of W/C_DA) are lower than in the intermediate-thickness region. In this intermediate-thickness region, the approximations upon which Eqs. (4) and (5) are based do not hold, but the actual value of T_w must lie below the values given by these approximations.

It is most interesting to note that even in the intermediate-thickness region the maximum calculated temperature is still below the melting point of steel, leading to the conclusion that *all* such disks would successfully re-enter the earth's atmosphere. For other materials or for other shapes (such as spheres, hollow spheres, etc.), the curves of maximum temperature vs a characteristic dimension of the body will be qualitatively similar to curves of T_w vs t given in Fig. 1 for steel disks. In other cases, however, an intermediate-size region may exist where the calculated temperatures exceed the melting point of the material, and the object may not re-enter successfully, as found for re-entry at meteor velocities.[7]

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Regions of Libration for a Symmetrical Satellite

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A SMALL, symmetrical rigid body moves in a circular orbit about a large point mass. The gravity gradient across the body, due to the attraction of the large point mass, produces a torque on the body about its mass center. Because of symmetry the component of angular velocity along the symmetry axis (the ζ axis) is a constant that will be taken to be zero. The positions of stable, relative equilibrium for ζ are a) ζ is aligned with the radius vector to the point mass when ζ is the axis of least inertia, and b) ζ is orthogonal to the radius vector to the point mass and the normal to the orbit plane when ζ is the axis of greatest inertia.

Motions of ζ which remain close to each of the equilibrium positions are called *librations*. A closed form solution for the librational motion is not available. However, the system does have an energy integral, which will be used to establish the regions of stable motion of ζ about the equilibrium positions. This approach follows G. W. Hill,[1] who used the energy integral in the restricted problem of three bodies to establish the region of motion for the moon.

Energy Integral

Let xyz be an orthogonal triad with origin at the mass center 0 of the body, z normal to the orbit plane, and x in the initial direction of the point mass P. Let ξ point continuously from 0 to P. The angle ν between ξ and x is equal to nt where n is the orbit rate and t is the time. The orientation of the symmetry axis ζ is defined by the angles ϕ and λ; ϕ is measured from ξ to the projection of ζ on the orbit plane and λ is measured from the orbit plane to ζ. The angle ψ specifies the rotation of the body about ζ and completes the Eulerian set ϕ, λ, and ψ.

The kinetic energy of the body about 0 is

$$T = \tfrac{1}{2}A[\dot{\lambda}^2 + (\dot{\phi} + n)^2 \cos^2\lambda] + \tfrac{1}{2}C[\dot{\psi} + (\dot{\phi} + n)\sin\lambda]^2$$

The gravity torque is derivable from the potential function

$$V = \tfrac{3}{2}n^2(C - A)\cos^2\lambda \cos^2\phi$$

Since neither T nor V depend explicitly on ψ, the system has the linear velocity integral

$$\dot{\psi} + (\dot{\phi} + n)\sin\lambda = s$$

where s, the spin rate, is the ζ component of the total angular velocity. The spin rate is a constant, which is taken as zero in this analysis.

The system also has an energy integral. Using the equation for $s = 0$ to eliminate $\dot{\psi}$, the energy integral becomes $R_2 - R_0 = \text{const}$ where

$$R_2 = \tfrac{1}{2}A(\dot{\lambda}^2 + \dot{\phi}^2\cos^2\lambda)$$

$$R_0 = -\tfrac{1}{2}An^2\sin^2\lambda - \tfrac{3}{2}(A - C)n^2(\sin^2\lambda + \cos^2\lambda\sin^2\phi)$$

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With $\nu = nt$ as the independent variable and after multiplying by $2/An^2$ the energy integral can be expressed as

$$\rho_2 - \rho_0 = k$$

where

$$\rho_2 = \lambda'^2 + \phi'^2 \cos^2\lambda$$

$$\rho_0 = -(3h + 1)\sin^2\lambda - 3h\cos^2\lambda \sin^2\phi$$

$$h = 1 - (C/A) \qquad k = \text{const}$$

and primes denote derivatives with respect to ν. For real bodies h is restricted to the regions $-1 < h < 1$, with $h = 0$ corresponding to a sphere.

Equilibrium Positions

The positions of stable relative equilibrium are

$$\left.\begin{aligned} E1&: \phi = \lambda = \phi' = \lambda' = 0 \\ E2&: \phi = \pi \qquad \lambda = \phi' = \lambda' = 0 \end{aligned}\right\} \quad \text{when } h > 0$$

$$\left.\begin{aligned} E3&: \phi = \pi/2 \qquad \lambda = \phi' = \lambda' = 0 \\ E4&: \phi = -\pi/2 \qquad \lambda = \phi' = \lambda' = 0 \end{aligned}\right\} \quad \text{when } h < 0$$

$E1$ and $E2$ correspond to the cases when ζ points toward and away from P, respectively. According to a theorem due to Liapunov,[2] $E1$ and $E2$ are stable if $\rho_2 - \rho_0$ is positive-definite in ϕ, λ, ϕ', and λ'. This condition is satisfied if, and only if, $h > 0$.

To verify that $E3$ and $E4$ are also stable equilibrium positions, the variable $\beta = \pi/2 - \phi$ is introduced. The energy integral can be put in the form $\sigma_2 - \sigma_0 = l$ where

$$\sigma_2 = \lambda'^2 + \beta'^2\cos^2\lambda \qquad \sigma_0 = -\sin^2\lambda + 3h\cos^2\lambda\sin^2\beta$$

and $l = k - 3h$. The function $\sigma_2 - \sigma_0$ is positive-definite in β, λ, β', and λ' if, and only if, $h < 0$. This insures the stability of $E3$ and $E4$.

Regions of Libration

The region of motion in $\phi\lambda$ space is $-\pi \le \phi \le \pi$ and $-\pi/2 \le \lambda \le \pi/2$. Imagine a unit sphere with origin at 0. As the body rotates about 0, the symmetry axis traces a path on the unit sphere with coordinates ϕ and λ. The lines defined by the equation $-\rho_0(\phi,\lambda) = k$ are contours of zero angular velocity of ζ in $\phi\lambda$ space. They are boundaries between real ($\rho_2 > 0$) and imaginary ($\rho_2 < 0$) motion.

The equation for the contours in Fig. 1 for $h > 0$ is

$$(3h + 1)\sin^2\lambda + 3h\cos^2\lambda\sin^2\phi = k$$

The contours are symmetric about $\lambda = 0$ and about $\phi = 0$

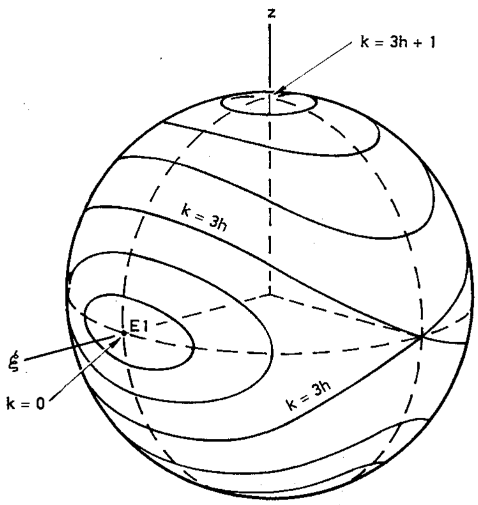

Fig. 1 Contours of zero velocity, $h > 0$

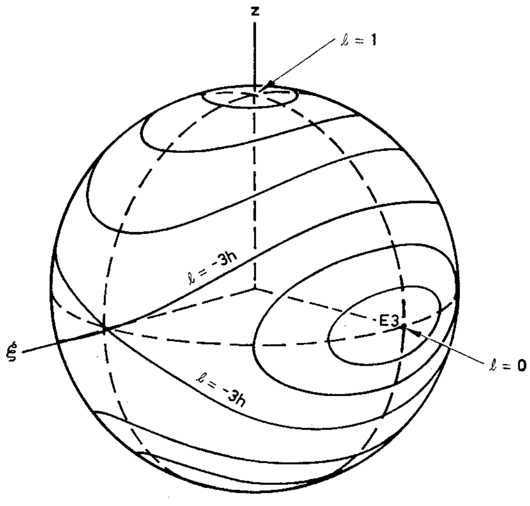

Fig. 2 Contours of zero velocity, $-\frac{1}{3} < h < 0$

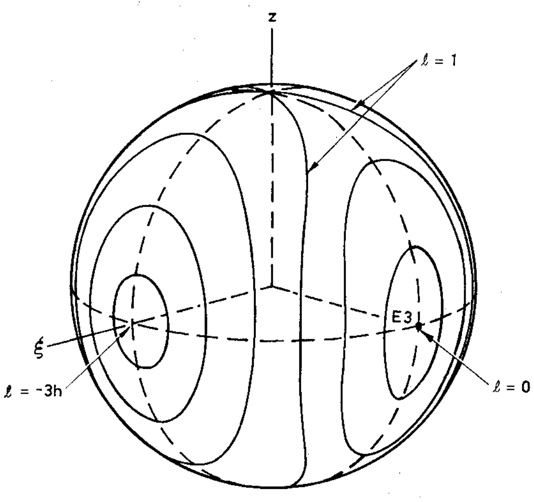

Fig. 3 Contours of zero velocity, $-1 < h < -\frac{1}{3}$

and π. For $k = 0$ they reduce to $E1$ and $E2$. For small values of k the contours are closed curves around $E1$ and $E2$, and the motion interior to the contours is real. The contours diverge from $E1$ and $E2$ as k increases. When the value $k = 3h$ is reached, the contours join at the two points $\lambda = 0$ and $\phi = \pm\pi/2$. The maximum absolute value

$$|\lambda| = \sin^{-1}[3h/(3h + 1)]^{1/2} < \pi/2$$

occurs on the meridians $\phi = 0$ and π. The reason the two regions interior to the $3h$ contours do not extend to the poles $\lambda = \pm\pi/2$ is that the $\phi\lambda$ space rotates about z at the orbit rate. Consequently, the body derives a measure of gyroscopic stability. The significance of the $3h$ contours is that a point initially near $E1$ and with $k < 3h$ can never approach $E2$. However, if the same point has a $k = 3h$, it can slip into the region near $E2$ which is interior to the $3h$ contours. Thus the $3h$ contours define the regions of stable motion about $E1$ and $E2$. These are called the regions of libration.

For $k > 3h$ the contours are closed curves around the poles, and the region between the two contours is real. The contours converge to the poles as k increases. When $k = 3h + 1$ the contours reduce to points at the poles. A point with $k > 3h + 1$ can move anywhere in $\phi\lambda$ space and will never come to rest in this space.

The equation for the zero velocity contours when $h < 0$ is

$$\sin^2\lambda - 3h\cos^2\lambda\sin^2\beta = l$$

The contours converge to $E3$ and $E4$ for decreasing values of

l. The following values are critical:

$$\sin^2\lambda = l \qquad \text{when } \beta = 0,\pi$$

$$\sin^2\beta = (-l/3h) \qquad \text{when } \lambda = 0,\pi$$

$$\sin^2\lambda = [(3h + l)/(3h + 1)] \qquad \text{when } \beta = \pm\pi/2$$

Thus for $-\frac{1}{3} < h < 0$ the contours appear as in Fig. 2 with the regions of libration interior to the $l = -3h$ contours. For $-1 < h < -\frac{1}{3}$ the contours appear as in Fig. 3 with the regions of libration interior to the $l = 1$ contours.

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Oscillations of a Fluid in a Rectilinear Conical Container

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In a recent publication, Troesch[1] notes that very few exact solutions are known for the free oscillations of an ideal (incompressible, nonviscous, irrotational) fluid in axially symmetric containers. He refers to the cylinder and paraboloid treated in Lamb;[2] however, the special case of a rectilinear conical container is attributed to Levin as an unpublished memorandum. Although the closed-form solution (for the first eigenvalue) now can be obtained quite easily from the comprehensive material presented by Troesch, it is of interest to record the initial special case that gave rise to the subsequent generalizations. Furthermore, Troesch's approach proceeds from a general series expansion followed by an examination of the results to determine what class of problems has been solved. The present method is direct in that the geometry of the given problem dictates the choice of a suitable coordinate system.

Consider a conical tank at rest with semivertex angle $\alpha = \pi/4$ as shown in Fig. 1. The linearized hydrodynamic equations are

$$\nabla^2\phi = 0 \text{ throughout the fluid} \tag{1}$$

$$\partial\phi/\partial n = 0 \text{ on the tank walls} \tag{2}$$

$$(\partial^2\phi/\partial t^2) + g(\partial\phi/\partial y) = 0 \text{ on the free surface } y = h \tag{3}$$

where ϕ is the scalar velocity potential. Using cylindrical coordinates [with $r = (x^2 + z^2)^{1/2}$] and a trial form for ϕ given by

$$\phi = F(t)G(r,y)\cos\theta \tag{4}$$

leads to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\partial^2 G}{\partial y^2} - \frac{G}{r^2} = 0 \tag{5}$$

The change of variables $W = r + y$ and $V = r - y$ results in coordinate surfaces for the tank walls. Equation (5) becomes

$$(W^2 + 2WV + V^2)[G_{WW} + G_{VV}] + (W + V)[G_W + G_V] - 2G = 0 \tag{6}$$

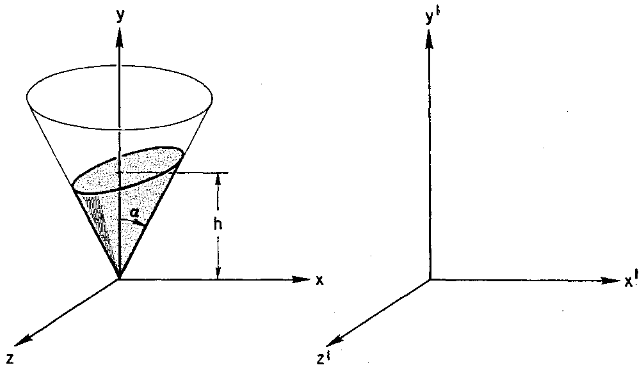

Fig. 1 Tank geometry

and condition (2) becomes

$$\partial G/\partial V = 0 \text{ on } V = 0 \tag{7}$$

A trial expansion for G in the form

$$G = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} W^m V^n$$

when substituted into Eq. (6) leads to double recursion relations for the A_{ij}. In order to satisfy these relations and Eq. (7), it is necessary that all coefficients A_{ij} vanish except $A_{20} = -A_{02}$; hence

$$G = A_{20}(W^2 - V^2) \qquad G = Kry \tag{8}$$

Thus,

$$\phi = F(t)ry\cos\theta \tag{9}$$

This solution fulfills conditions (1) and (2). Condition (3) leads to

$$hr\ddot{F}(t) + rgF(t) = 0$$
$$F(t) = A\cos(\omega t + \lambda) \tag{10}$$

where $\omega = (g/h)^{1/2}$ represents the (first) natural frequency of the system.

If the tank is not at rest but subjected to rectilinear accelerations, condition (3) is replaced by

$$(\partial^2\phi/\partial t^2) - A_y(\partial\phi/\partial y) + \dot{A}_x x = 0 \text{ on the free surface } y = h \tag{3'}$$

where A_x and A_y are the accelerations of the tank with respect to an inertial coordinate system (x',y',z'). The time dependence of the velocity potential now is given by

$$h\ddot{F}(t) - A_y(t)F(t) + \dot{A}_x(t) = 0$$

For example, a periodic forced excitation in the x direction leads to

$$h\ddot{F}(t) + gF(t) = B\cos\alpha t \qquad \alpha \neq \omega$$

$$F = A\cos(\omega t + \lambda) + [B\cos\alpha t/(g - h\alpha^2)]$$

The first term actually is damped for a real fluid, and consequently the motion is represented by the second term. The resonance as $\alpha \to \omega = (g/h)^{1/2}$ has been verified experimentally.

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